Example VAR(1), 2 variables, with Cholesky identification

1st order

- $\mathbf{y}_t^\star = \left[\begin{array}{cc} x_t & y_t \end{array} \right]'$, $\mathbf{u}_t = \left[\begin{array}{cc} u_t^x & u_t^y \end{array} \right]'$
- reduced form after the estimation of coefficients (0.2, 0.5, 0.1, 0.7) using time series x and y

$$\begin{aligned} x_t &= 0.2x_{t-1} + 0.5y_{t-1} + u_t^x \\ y_t &= 0.1x_{t-1} + 0.7y_{t-1} + u_t^y \end{aligned}$$

  in matrix notation (knowing that it comes from a SVAR)

$$\mathbf{y}_t^\star = \underbrace{\left[\begin{array}{cc} 0.2 & 0.5 \\ 0.1 & 0.7 \end{array} \right]}_{\substack{\mathsf{A}_1 = \\ \mathbf{A}_0^{-1}\mathbf{A}_1 = \\ \mathbf{B}\mathbf{A}_1}} \mathbf{y}_{t-1}^\star + \underbrace{\mathbf{u}_t}_{\mathbf{B}\mathbf{e}_t}$$

  after the estimation, time series u^x and u^y are known; calculate their variance-covariance matrix

$$\mathtt{Var}\mathbf{u}_t = \left[\begin{array}{cc} \sigma_{u^x}^2 & \sigma_{u^y u^x} \\ \sigma_{u^x u^y} & \sigma_{u^y}^2 \end{array} \right] = \left[\begin{array}{cc} 2 & -1.6 \\ -1.6 & 5.28 \end{array} \right]$$

- knowing that $\mathbf{B}\mathbf{e}_t = \mathbf{u}_t$, calulate variance

$$\mathtt{Var}\mathbf{u}_t = \mathtt{Var}\left(\mathbf{B}\mathbf{e}_t\right) = \mathbf{B}\left(\mathtt{Var}\mathbf{e}_t\right)\mathbf{B}' = \mathbf{B}\left[\begin{array}{cc} \sigma_{e^x}^2 & 0 \\ 0 & \sigma_{e^y}^2 \end{array} \right]\mathbf{B}'$$

Cholesky decomposition of the variance-covariance matrix

$$\begin{bmatrix} 2 & -1.6 \\ -1.6 & 5.28 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ -0.8 & 1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}}_{\mathtt{Var}\mathbf{e}_t} \underbrace{\begin{bmatrix} 1 & -0.8 \\ 0 & 1 \end{bmatrix}}_{\mathbf{B}'}$$

- SVAR is now identified, in matrix form (to get $\mathbf{A}_1$, use $\mathsf{A}_1 = \mathbf{B}\mathbf{A}_1 \rightarrow \mathbf{A}_0\mathsf{A}_1 = \mathbf{A}_1$)

$$\begin{aligned} \mathbf{A}_0\mathbf{y}_t^\star &= \mathbf{A}_1\mathbf{y}_{t-1}^\star + \mathbf{e}_t \\ \begin{bmatrix} 1 & 0 \\ 0.8 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} &= \begin{bmatrix} 0.2 & 0.5 \\ 0.26 & 1.1 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} e_t^x \\ e_t^y \end{bmatrix} \end{aligned}$$

or in equation form

$$\begin{aligned} x_t &= 0.2x_{t-1} + 0.5y_{t-1} + e_t^x \\ y_t &= -0.8x_t + 0.26x_{t-1} + 1.1y_{t-1} + e_t^y \end{aligned}$$

2nd order

- $\tilde{\mathbf{y}}_t^\star = \begin{bmatrix} y_t & x_t \end{bmatrix}'$, $\tilde{\mathbf{u}}_t = \begin{bmatrix} u_t^y & u_t^x \end{bmatrix}'$
- reduced form after the estimation of coefficients (0.2, 0.5, 0.1, 0.7) using time series x and y

$$\begin{aligned} x_t &= 0.2x_{t-1} + 0.5y_{t-1} + u_t^x \\ y_t &= 0.1x_{t-1} + 0.7y_{t-1} + u_t^y \end{aligned}$$

in matrix notation (knowing that it comes from a SVAR)

$$\tilde{\mathbf{y}}_t^\star = \underbrace{\begin{bmatrix} 0.1 & 0.7 \\ 0.2 & 0.5 \end{bmatrix}}_{\substack{\tilde{\mathsf{A}}_1 = \\ \tilde{\mathbf{A}}_0^{-1}\tilde{\mathbf{A}}_1 = \\ \tilde{\mathbf{B}}\tilde{\mathbf{A}}_1}} \tilde{\mathbf{y}}_{t-1}^\star + \underbrace{\tilde{\mathbf{u}}_t}_{\tilde{\mathbf{B}}\tilde{\mathbf{e}}_t}$$

after the estimation, time series u^x and u^y are known; calculate their variance-covariance matrix

$$\texttt{Var}\tilde{\mathbf{u}}_t = \begin{bmatrix} \sigma^2_{u^y} & \sigma_{u^x u^y} \\ \sigma_{u^y u^x} & \sigma^2_{u^x} \end{bmatrix} = \begin{bmatrix} 5.28 & -1.6 \\ -1.6 & 2 \end{bmatrix}$$

- ☐ knowing that $\tilde{\mathbf{B}}\tilde{\mathbf{e}}_t = \tilde{\mathbf{u}}_t$, calulate variance

$$\texttt{Var}\tilde{\mathbf{u}}_t = \texttt{Var}\left(\tilde{\mathbf{B}}\tilde{\mathbf{e}}_t\right) = \tilde{\mathbf{B}}\left(\texttt{Var}\tilde{\mathbf{e}}_t\right)\tilde{\mathbf{B}}' = \tilde{\mathbf{B}}\begin{bmatrix} \sigma^2_{e^y} & 0 \\ 0 & \sigma^2_{e^x} \end{bmatrix}\tilde{\mathbf{B}}'$$

Cholesky decomposition

$$\begin{bmatrix} 5.28 & -1.6 \\ -1.6 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ -0.3 & 1 \end{bmatrix}}_{\tilde{\mathbf{B}}}\underbrace{\begin{bmatrix} 5.28 & 0 \\ 0 & 1.52 \end{bmatrix}}_{\texttt{Var}\tilde{\mathbf{e}}_t}\underbrace{\begin{bmatrix} 1 & -0.3 \\ 0 & 1 \end{bmatrix}}_{\tilde{\mathbf{B}}'}$$

- ☐ SVAR is now identified, in matrix form (to get $\tilde{\mathbf{A}}_1$, use $\tilde{\mathsf{A}}_1 = \tilde{\mathbf{B}}\tilde{\mathbf{A}}_1 \rightarrow \tilde{\mathbf{A}}_0\tilde{\mathsf{A}}_1 = \tilde{\mathbf{A}}_1$)

$$\begin{aligned} \tilde{\mathbf{A}}_0\tilde{\mathbf{y}}^\star_t &= \tilde{\mathbf{A}}_1\tilde{\mathbf{y}}^\star_{t-1} + \tilde{\mathbf{e}}_t \\ \begin{bmatrix} 1 & 0 \\ 0.3 & 1 \end{bmatrix}\begin{bmatrix} y_t \\ x_t \end{bmatrix} &= \begin{bmatrix} 0.1 & 0.7 \\ 0.23 & 0.71 \end{bmatrix}\begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} e^y_t \\ e^x_t \end{bmatrix} \end{aligned}$$

or in equation form

$$\begin{aligned} y_t &= 0.7x_{t-1} + 0.1y_{t-1} + e^y_t \\ x_t &= -0.3y_t + 0.71x_{t-1} + 0.23y_{t-1} + e^x_t \end{aligned}$$

Comparison

- ☐ 1st order

$$\begin{aligned} x_t &= 0.2x_{t-1} + 0.5y_{t-1} + e^x_t \\ y_t &= -0.8x_t + 0.26x_{t-1} + 1.1y_{t-1} + e^y_t \end{aligned}$$

- 2nd order

$$\begin{aligned} x_t &= -0.3y_t + 0.71x_{t-1} + 0.23y_{t-1} + e_t^x \\ y_t &= 0.7x_{t-1} + 0.1y_{t-1} + e_t^y \end{aligned}$$

- completely different models implying completely different dynamics (i.e. IRF)

Pro memoria: matrix algebra

- Size matters, conformable matrices
- Multiplication

$$\begin{bmatrix} a & b \end{bmatrix} \times \begin{bmatrix} c \\ d \end{bmatrix} = ac + bd$$

- Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

- Inverse

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \begin{bmatrix} 1 & 0 & \dots \\ 0 & 1 & \\ \vdots & & \ddots \end{bmatrix}$$

- Premultiplication, postmultiplication, multiplication is not commutative

$$\mathbf{AB} \neq \mathbf{BA}$$